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THE VELOCITY AND TEMPERATURE DISTRIBUTION IN ONE-DIMENSIONAL FLOW WITH TURBULENCE AUGMENTATION AND PRESSURE GRADIENT

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Abstract—The Kolmogorov–Prandtl turbulence energy hypothesis is formulated in a way which is valid for the laminar sublayer as well as the fully turbulent region of a one-dimensional flow. The necessary constants are fitted to available experimental data. Numerical solutions are obtained for Couette flow with turbulence augmentation and pressure gradient and for turbulent duct flow. Reasonable agreement with available experimental data is obtained. Some new dimensionless groups are used and shown to be superior to the ones based on the friction velocity. The effects of turbulence augmentation and pressure gradient on the velocity and temperature distribution are studied. It is found that the solutions tend to approach solutions for limiting cases. The results are plotted in some figures in Section 5.

NOMENCLATURE

- A_D, a constant in the equation for length scale of dissipation;
- A_μ, a constant in the equation for length scale of viscosity;
- C_D, the dissipation constant in the turbulence energy equation;
- C_μ, the turbulent viscosity constant;
- D, the dissipation of turbulence energy;
- J, flux;
- k, turbulence energy;
- l_D, length scale of dissipation;
- l_μ, length scale of viscosity;
- p', pressure gradient parallel to the wall;
- R, Reynolds number $\equiv (\sqrt{(k)}y\rho/\mu)$;
- s, friction coefficient $\equiv \tau_S y/\mu u$;
- S, Stanton number $\equiv \sigma J_S y/\mu\phi$;
- u, mean velocity parallel to the wall;
- u', v', w', fluctuating velocities;
- y, distance normal to the wall.

Greek symbols

- Γ, conserved property transport coefficient;
- η, the dimensionless turbulent viscosity;
- ε, $\equiv \mu_t/\mu$;
- μ, the laminar viscosity;
- ρ, the density;
- σ, the Prandtl number;
- τ, the shear stress;
- φ, a conserved property.

Subscripts

- e, effective;
- G, at the outer edge of the layer;
- k, for the turbulence energy;
- S, on the wall;
- t, turbulent;
- $+$, dimensionless quantities based on the friction velocity.

1. INTRODUCTION

THIS paper deals with the influence of turbulence on the flow near solid walls, when the state of the fluid at any point can be expressed as a function of the distance from the wall only. The investigation of such flows constitutes an important stage in the development of solution procedures for two-dimensional turbulent flows.

Firstly because in this mathematically simpler configuration it is easier to formulate turbulent viscosity hypotheses, and compare their implications with experimental data. Secondly because we can save a considerable amount of computer time in the computation of two-dimensional flows, if we employ one-dimensional solutions in the vicinity of solid walls, where, due to the existence of a boundary-layer, a large number of mesh points is otherwise necessary (Wolfshtein [1]).

Another advantage of the present investigation is that one-dimensional flows are frequently met in engineering. The turbulent "logarithmic law of the wall" is the most common of such flows, but in fact all fully developed flows in plane or axially-symmetrical ducts with uniform cross section are one-dimensional; and in many cases important regions of more complex flows are very nearly one-dimensional as well.

In the past the Prandtl mixing-length hypothesis was extensively used in treatment of such flows. Its application in the fully turbulent region results in the logarithmic law of the wall. Van-Driest [2] suggested a way to extend the mixing-length hypothesis to the laminar sublayer, and Patankar [3] used an extended form of van-Driest's proposal to obtain a generalised solution of one-dimensional flows. However, the mixing-length hypothesis, even in its most developed form, suffers from a number of drawbacks, the most important of them are:

(i) It is valid only when local equilibrium exists between generation and dissipation of turbulence.

(ii) It implies zero turbulent exchange coefficients in regions of zero velocity gradients.

(iii) It has not been proven to be an effective tool in two-dimensional situations.

Obviously, the above limitations are a direct consequence of the inflexibility of the mixing-length hypothesis, by which the state of turbulence of the fluid is assumed to be dependent on the velocity field, and one additional quantity, the mixing length. The mixing length is usually assumed to be related to the geometry of the flow. Therefore we can not prescribe the turbulence level, nor investigate turbulence augmentation when we use the mixing-length hypothesis.

In the present paper the Kolmogorov [4] Prandtl [5] turbulence-energy hypothesis will be used. In this hypothesis the local state of turbulence of the fluid is assumed to depend on a length scale and on the kinetic energy of the turbulent velocity fluctuations.* The hypothesis has already been used for the solution of one-dimensional and boundary-layer problems in a number of cases. Emmons [6] presented the hypothesis in a very straightforward manner, and obtained some solutions for flows away from walls. Glushko [7] used two different length scales for turbulence generation and dissipation near walls, but his expressions were somewhat obscure. He did not try to investigate turbulence augmentation. Spalding [8, 9] tried to obtain analytical solutions by the use of a distinct boundary between the laminar sublayer and the fully turbulent region, and a discontinuous eddy viscosity.

The purpose of the present paper is two-fold: Firstly, to present the implications of the hypothesis, for one-dimensional flow, in a convenient and general form, applicable to both the laminar sublayer and the fully turbulent region. Secondly, to study the effects of turbulence augmentation on one-dimensional flows.

The present paper is restricted to steady incompressible turbulent flow with uniform properties. We shall be concerned with the three second order differential equations for the velocity, u, a conserved property, φ, and the turbulence energy, k. Of these, the first two may be analytically integrated once. The second order turbulence energy equation will be solved by a numerical iterative method. The two first

* An implication of this hypothesis is that the turbulent viscosity is a scalar. This is true in one-dimensional flows, but not in two- and three-dimensional ones. However, the hypothesis is usually assumed to be a reasonable approximation also in two-dimensional flows.

order equations may be numerically integrated without iterations. Details of all these operations are described in Section 2. Section 3 is concerned with some analytical solutions for limiting cases, and in Section 4 numerical values are assigned to all the necessary empirical constants. The results of computations of Couette and duct flow are described in Section 5. The influence of turbulence augmentation and pressure gradient on the velocity and conserved property distributions in a Couette flow is studied, and comparison with recent duct flow data is made. Further discussion and conclusions are presented in Section 6.

2. THE MATHEMATICAL FORMULATION

The conservation equations

We consider a flow parallel to a wall, where all the quantities are not varying in the directions parallel to the wall. The treatment is restricted to steady, uniform-property incompressible flow. We wish to predict the time-averaged velocity u, a time-averaged conserved property φ, and the turbulence energy k, which is defined as

$$k \equiv \tfrac{1}{2}(\overline{u'^2} + \overline{u'^2} + \overline{w'^2}). \tag{2.1}$$

All these quantities are dependent on the distance from the wall, y, and the fluid properties. The governing equations do not contain any convective terms, and may be shown to be:

$$\mu\frac{d^2u}{dy^2} + \frac{d\tau_t}{dy} - p' = 0 \tag{2.2}$$

$$\Gamma\frac{d^2\varphi}{dy^2} + \frac{dJ_t}{dy} = 0 \tag{2.3}$$

$$\mu\frac{d^2k}{dy^2} + \frac{dJ_{k,t}}{dy} + \tau_t\frac{du}{dy} - D = 0 \tag{2.4}$$

where J_t and $J_{k,t}$ are the "diffusional" fluxes of φ and k respectively due to the turbulent fluctuations; τ_t is the Reynolds stress; p' is the pressure gradient parallel to the wall; D is the dissipation of turbulence energy into heat; μ and Γ are the laminar transport coefficients for momentum and conserved property respectively. The boundary conditions for these equations are:

$$\text{at} \quad y = 0 \qquad u = \varphi = k = 0;$$

$$\mu\frac{du}{dy} = \tau_S; \quad \alpha\frac{d\varphi}{dy} = J_S \tag{2.5}$$

$$\text{at} \quad y = y_G \qquad k = k_G \tag{2.6†}$$

where τ_S and J_S are the skin friction and wall φ-flux respectively and k_G is given. The value of y_G will be always large enough to ensure that a part of the fully turbulent‡ region is included in the integration.

The turbulent quantities

In order to obtain a solution to equations (2.2), (2.3) and (2.4), we must relate the quantities τ_t, J_t, $J_{k,t}$ and D to u, φ and k. To do this we shall use the Kolmogorov-Prandtl model of turbulence. We shall sum here the implications of this model, as presented by Wolfshtein [1]:

$$\tau_t = \mu_t\frac{du}{dy} \tag{2.7}$$

$$J_t = \frac{\mu_t}{\sigma_t}\frac{d\varphi}{dy} \tag{2.8}$$

$$J_{k,t} = \frac{\mu_t}{\sigma_{k,t}}\frac{dk}{dy} \tag{2.9}$$

$$\mu_t = c_\mu \rho k^{\frac{1}{2}} l_\mu \tag{2.10}$$

$$D = \frac{C_D \rho k^{\frac{3}{2}}}{l_D} \tag{2.11}$$

where σ_t, $\sigma_{k,t}$, C_μ, C_D are empirical constants, and l_μ and l_D are the length scales for turbulent diffusion and dissipation respectively.

It is more convenient to define effective transport properties, as follows:

$$\mu_e \equiv \mu + \mu_t \tag{2.12}$$

$$\Gamma_e \equiv \frac{\mu}{\sigma} + \frac{\mu_t}{\sigma_t} \tag{2.13}$$

† In a duct flow equation (2.6) is replaced by

$$\text{at } y = y_G \ \frac{dk}{dy} = 0.$$

‡ The fully turbulent region is this region where the laminar transport properties have no influence on the flow.

$$\Gamma_{k,e} \equiv \mu + \frac{\mu_t}{\sigma_{k,t}} \tag{2.14}$$

where σ is the Prandtl number.

By substitution of all these relations in equations (2.2), (2.3) and (2.4), and integration of equations (2.2) and (2.3) we get

$$\mu_e \frac{\mathrm{d}u}{\mathrm{d}y} = \tau_S + p'y \tag{2.15}$$

$$\Gamma_e \frac{\mathrm{d}\varphi}{\mathrm{d}y} = J_S \tag{2.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\Gamma_{k,e}\frac{\mathrm{d}k}{\mathrm{d}y}\right) + \mu_t\left(\frac{\tau_S + p'y}{\mu_e}\right)^2 - \frac{C_D\rho k^{\frac{3}{2}}}{l_D} = 0. \tag{2.17}$$

The turbulence length scales

We shall not, at present, write a differential equation for l_μ and l_D. Instead, we shall devise empirical functions to describe them. Near solid walls both are known to be proportional to y. It was also suggested (e.g. by Glushko [7] and Spalding [8]) that in the laminar sublayer both these quantities should be proportional to $R \,.\, y$, where R, the Reynolds number of turbulence, is defined as

$$R \equiv \frac{k^{\frac{1}{2}} y\rho}{\mu}. \tag{2.18}$$

An examination of equations (2.4), (2.10) and (2.11) reveals that, without any loss of generality, we may choose the empirical coefficients C_μ and C_D in such a way that, in the fully turbulent region

$$l_\mu = l_D = y \tag{2.19}$$

l_μ is proportional to l_D also in the laminar sublayer, but we cannot expect them to be equal there. Therefore, in the laminar sublayer

$$l_\mu = A_\mu R y \tag{2.20}$$

$$l_D = A_D R y. \tag{2.21}$$

Information about the intermediate region where both equation (2.19) and equation (2.20) or (2.21) are inaccurate is very scanty. However, l_μ should have some similarity to the Prandtl mixing length. Therefore we shall use expressions similar to that proposed by van-Driest [4] for the mixing length, i.e.

$$l_\mu = y[1 - \exp(-A_\mu R)] \tag{2.22}$$

$$l_D = y[1 - \exp(-A_D R)]. \tag{2.23}$$

These two expressions satisfy equations (2.19), (2.20) and (2.21). They are likely to be a fair approximation of l_μ and l_D also in the transition region between the laminar sublayer and fully turbulent layer.

Non-dimensional quantities

Equations (2.15), (2.16) and (2.17) may be non-dimensionalised by the use of the following dimensionless groups:

$$y_+ \equiv \frac{y\sqrt{\tau_S\rho}}{\mu} \tag{2.24}$$

$$u_+ \equiv u\sqrt{\frac{\rho}{\tau_S}} \tag{2.25}$$

$$\varphi_+ \equiv \frac{\varphi\sqrt{\tau_S\rho}}{J_S} \tag{2.26}$$

$$k_+ \equiv \frac{k\rho}{\tau_S} \tag{2.27}$$

$$p_+ \equiv \frac{p'\mu}{\sqrt{(\rho\tau_S)}\,\tau_S} \tag{2.28}$$

$$\varepsilon \equiv \frac{\mu_t}{\mu} \tag{2.29}$$

$$L_\mu \equiv l_\mu/y \tag{2.30}$$

$$L_D \equiv l_D/y. \tag{2.31}$$

The equations then are

$$(1 + \varepsilon)\frac{\mathrm{d}u_+}{\mathrm{d}y_+} = 1 + p_+ y_+ \tag{2.32}$$

$$\left(\frac{1}{\sigma} + \frac{\varepsilon}{\sigma_t}\right)\frac{\mathrm{d}\varphi_+}{\mathrm{d}y_+} = 1 \tag{2.33}$$

$$\frac{d}{dy_+}\left[\left(1+\frac{\varepsilon}{\sigma_{k,t}}\right)\frac{dk_+}{dy_+}\right]+\varepsilon\left(\frac{1+p_+y_+}{1+\varepsilon}\right)^2 - C_D\frac{k_+^{\frac{3}{2}}}{L_D y_+}=0 \quad (2.34)$$

$$\varepsilon = C_\mu L_\mu(\sqrt{k_+})\, y_+ \quad (2.35)$$

with the following conditions

$$\text{at } y_+=0 \quad u_+=\varphi_+=k_+=0 \quad (2.36)$$

$$\text{at } y_+=y_{G+} \quad k_+=k_{G+} \quad (2.37)$$

$$L_\mu = 1-\exp\left[-A_\mu(\sqrt{k_+})\, y_+\right] \quad (2.38)$$

$$L_D = 1-\exp\left[-A_D(\sqrt{k_+})\, y_+\right]. \quad (2.39)$$

It should be noted that many of the above quantities contain τ_S. When τ_S vanishes, y_+ and φ_+ will, therefore, vanish as well, while u_+ and k_+ will become infinite. In order to avoid such circumstances, it is preferable to present the results in terms of the following dimensionless groups

$$R \equiv y_+\sqrt{k_+} \quad (\text{replacing } y_+) \quad (2.40)$$

$$s \equiv y_+/u_+ \quad (\text{replacing } u_+) \quad (2.41)$$

$$S \equiv \sigma y_+/\varphi_+ \quad (\text{replacing } \varphi_+). \quad (2.42)$$

k_+ may not be altered in this way, and we shall continue to use it. It will be seen in further sections, however, that this set is satisfactory even when τ_S vanishes.

Solution method

To sum up, we wish to present k_+, s and S as functions of R, and of the two parameters p_+ and k_{+G}. Now, equations (2.32) and (2.33) may be very easily integrated. However, we need to solve equation (2.34) first. The equation may be solved by finite difference technique. A typical section of the mesh is shown in Fig. 1. The finite-difference counterpart of equation (2.34) may be written as

$$Dk_{+,P} = A^k_{+,E} + Bk_{+,W} + C \quad (2.43)$$

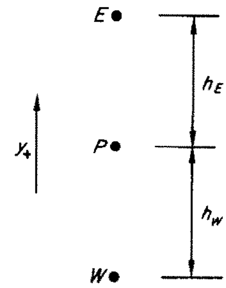

FIG. 1. The finite-difference mesh.

where

$$A = \frac{2+\dfrac{1}{\sigma_{k,t}}(\varepsilon_E+\varepsilon_P)}{(h_E+h_W)\,h_E} \quad (2.44)$$

$$B = \frac{2+\dfrac{1}{\sigma_{k,t}}(\varepsilon_W+\varepsilon_P)}{(h_E+h_W)\,h_W} \quad (2.45)$$

$$C = \varepsilon_P\left[\left(\frac{1+p_+y_+}{1+\varepsilon}\right)_P\right]^2 \quad (2.46)$$

$$D = A+B+C_D\left(\frac{\sqrt{k_+}}{L_D y_+}\right)_P. \quad (2.47)$$

The numerical solution of equation (2.43) does not present any difficulties. It was solved by a successive elimination process, described in [10], p. 97.

3. SOME ANALYTICAL SOLUTIONS

In general, equation (2.34) cannot be solved analytically. However, we may obtain analytical solutions to some limiting cases, and they are worth studying.

The non-diffusional layer

When y becomes very large, and the pressure gradient vanishes, we may write

$$\mu_e = C_\mu\rho(\sqrt{k})\, y \gg \mu \quad (3.1)$$

$$l_\mu = l_D = y \quad (3.2)$$

$$\Gamma_{k,t} = \frac{C_\mu}{\sigma_{k,t}}(\sqrt{k})\, y\rho. \quad (3.3)$$

In these conditions, a particular solution of equation (2.34) is

$$k_+ = (C_\mu C_D)^{-\frac{1}{2}}. \tag{3.4}$$

The solutions of equations (2.32) and (2.33) may then be written as

$$s = \left(\frac{C_\mu}{C_D}\right)^{\frac{1}{2}} \frac{R}{\ln\left[ER(C_\mu C_D)^{\frac{1}{4}}\right]} \tag{3.5}$$

$$S = \frac{\sigma/\sigma_0 (C_\mu C_D)^{\frac{1}{4}} R}{P + C_D^{\frac{1}{4}} C_\mu^{-\frac{3}{4}} \ln\left[E(C_\mu C_D)^{\frac{1}{4}} R\right]} \tag{3.6}$$

where E is an integration constant, and P is the resistance of the laminar-sublayer to φ-transfer, described by Spalding and Jayatillaka [11].

The linear shear layer

When y becomes very large and the skin friction vanishes, equations (3.1), (3.2) and (3.3) still hold. Spalding [9] showed that in this case the solution to equations (2.34) and (2.32) is

$$k_+ = \frac{(p_+ R)^{\frac{2}{3}}}{\left[C_\mu\left(C_D - \dfrac{2C_\mu}{3\sigma_{k,t}}\right)\right]^{\frac{1}{3}}} \tag{3.7}$$

$$\frac{1}{s} = \frac{2p_+^{\frac{2}{3}}}{C_\mu R^{\frac{1}{3}}}\left[C_\mu\left(C_D - \frac{2C_\mu}{3\sigma_{k,t}}\right)\right]^{\frac{11}{12}} + \text{Const}\,\frac{p_+^{\frac{1}{3}}}{R^{\frac{1}{6}}\left[C_\mu\left(C_D - \dfrac{2C_\mu}{3\sigma_{k,t}}\right)\right]^{\frac{1}{6}}}. \tag{3.8}$$

The laminar sublayer

In the laminar sublayer, where y is very small,

$$\mu_{\text{eff}} = \mu \gg \mu_t \tag{3.9}$$

and

$$l_\mu = A_\mu R y \tag{2.20}$$

$$l_D = A_D R y. \tag{2.21}$$

Therefore equation (2.34) reduces to:

$$\frac{\mathrm{d}^2 k_+}{\mathrm{d}y_+^2} = \frac{C_D}{A_D}\frac{k_+}{y_+^2}. \tag{3.10}$$

The solution of this equation is

$$k_+ = A R^{\frac{2B}{2+B}} \tag{3.11}$$

where A is an integration constant,† and

$$B = \frac{\sqrt{(4\,C_D/A_D + 1)} + 1}{2}. \tag{3.12}$$

It also follows that

$$\varepsilon = C_\mu A_\mu A^{\frac{2}{2-B}} (y_+)^{\frac{4}{2-B}} \tag{3.13}$$

$$\frac{1}{s} = 1 + \frac{p_+}{2}\frac{R}{\sqrt{k_+}} \tag{3.14}$$

$$S = 1. \tag{3.15}$$

The no-generation layer

In a fully turbulent layer, and when the shear stress is small, equation (2.34) reduces to

$$\frac{C_\mu}{\sigma_{k,t} C_D}\frac{\mathrm{d}}{\mathrm{d}y_+}\left(k_+^{\frac{1}{2}} y_+ \frac{\mathrm{d}k_+}{\mathrm{d}y_+}\right) = \frac{k_+^{\frac{3}{2}}}{y_+}. \tag{3.16}$$

Spalding [9] had shown that the solution of this equation, for large values of y_+ is

$$k_+ = a y_+^m \tag{3.17}$$

where

$$m = \left(\frac{2C_D \sigma_{k,t}}{3C_\mu}\right)^{\frac{1}{2}} \tag{3.18}$$

and a is an arbitrary constant. It may be easily shown that

$$k_+ = \text{const} \times R^{\frac{2m}{2+m}} \tag{3.19}$$

$$s = \text{const} \times R^{\frac{2}{2+m}} \tag{3.20}$$

$$S = \text{const} \times s. \tag{3.21}$$

4. EVALUATION OF THE EMPIRICAL CONSTANTS

In the presentation of the problem (section 2) we defined five empirical constants associated with the turbulence energy equation, namely

† The second integration constant is zero because a negative value for B is non-realistic.

C_μ, C_D, $\sigma_{k,t}$, A_μ and A_D; we also used one empirical constant $\sigma_{\varphi,t}$ in the conserved property equation. Obviously these constants must be determined from experimental results. However, there may be, in principle at least, more than six different sources of appropriate data, and our success to correlate with the present hypothesis as many of these data as possible will be also a measure of the validity of the hypothesis. Such a critical review is beyond the scope of the present paper, mainly because the amount of reliable experimental data available is very limited. Instead, the author tried to demonstrate that once a set of data is available the constants may be fixed fairly simply and reproduction of experimental results is possible then. For this purpose the following data was used:

(i) The constants E and $\varkappa$ in the logarithmic law of the wall (see Schlichting [12])

$$\varkappa u_+ = \ln(Ey_+) \tag{4.1}$$

where $\varkappa = 0{\cdot}4$, $E = 9$.

(ii) The constant K_0 in the veloicity profile in the linear shear layer

$$u = \frac{2}{K_0}\sqrt{\left(\frac{p'y}{\rho}\right)} + \text{const.} \tag{4.2}$$

This constant was recommended by Townsend [13] as

$$K_0 = 0{\cdot}48.$$

(iii) The constants a and α in the turbulent viscosity profile for the laminar sublayer:

$$\frac{\mu_t}{\mu} = a(y_+)^\alpha. \tag{4.3}$$

These may be deduced from a paper by Spalding and Jayatillaka [11] as $a = 8{\cdot}85 \times 10^{-5}$ and $\alpha = 4$.

(iv) The empirical P function describing the resistance of the laminar sublayer to heat transfer in a uniform-shear layer. This function was reported by Spalding and Jayatillaka [11]. The relation (iii) is derived from the asymptotic relation for the "P"-function at high Prandtl numbers, but we may still use the P-function correlation at low Prandtl numbers.

Summing up all the above data, we note that they are not known accurately and without any doubt. However, we shall use them to obtain tentative values of the constants used in the theory. When the data of (i) to (iv) is used, a good fit is obtained for the following values of the constants

$$C_\mu = 0{\cdot}220$$
$$C_D = 0{\cdot}416$$
$$\sigma_{k,t} = 1{\cdot}53$$
$$A_\mu = 0{\cdot}016$$
$$A_D = 0{\cdot}263$$
$$\sigma_{\varphi,t} = 0{\cdot}9.$$

Equations (4.2) and (4.3) are automatically satisfied by these values. The fit of the logarithmic law of the wall and the P-function are shown in Figs. 2 and 3, respectively, and is good.

It is of interest to compare these constants with those deduced from earlier work. We shall compare the results to those recommended by Wieghardt in an appendix to Prandtl's [5] paper, those recommended by Glushko [7], and two proposals by Spalding [8, 9], as given in Table 1.

Table 1

	Wieghardt [5]	Glushko [7]	Spalding [8]	Spalding [9]	Present work
C_μ	0·224	0·2	0·2	0·179	0·22
C_D	0·45	0·313	0·313	0·224	0·416
$\sigma_{k,t}$	1·47	2·5	1·7	2·13	1·53
A_μ		0·0091		0·0315	0·016
A_D		0·080		0·112	0·263

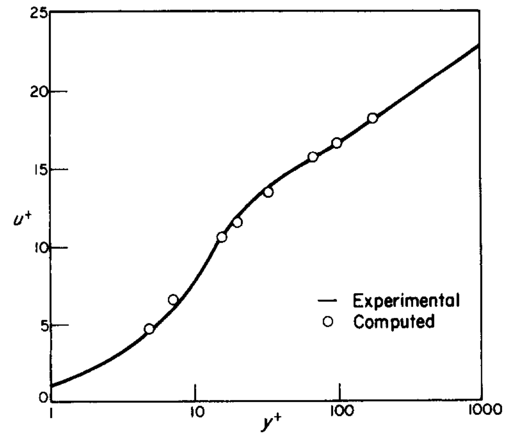

FIG. 2. Computed and measured velocity profile in a uniform-shear no-diffusion Couette flow. Data reported by Schlichting.

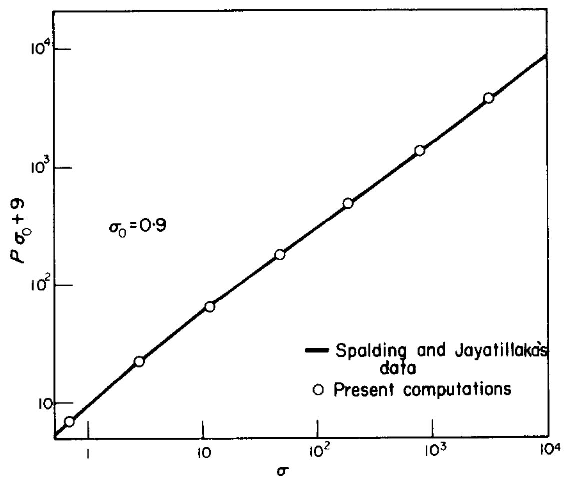

FIG. 3. Comparison of the P-function in a uniform-shear no-diffusion Couette flow with Spalding and Jayatillaka's recommendations.

The differences in Table 1 seem to have originated from two causes: first, that different sets of experimental data were used, and second that the integration method, the length scale distribution, and the methods for the choice of optimum values were different in each work. Once that more experimental data becomes available it will become necessary to repeat the above process of constant fitting. However, the author believes that the present method of constant-fitting is superior to other ones because it does not involve any mathematical simplifications, and it can accommodate any necessary physical hypothesis.

Another interesting comparison is that of the length scales distribution near the wall. In Fig. 4 the present l_μ and l_D distributions are compared with the implications of previous suggestions by Glushko [7] and Spalding [8]. There are differences between the three suggestions, which may be explained, at least partially, by the different hypotheses and by differences in the experimental data used by each of the three authors. The only way to determine the true values of the constants in equations (2.22) and (2.23), is by reference to more experimental data, when it becomes available.

It would be helpful to sum up what we have achieved until this point. An equation for the turbulence energy in one-dimensional flow was formulated, which is valid in the laminar sublayer, as well as in the transition and turbulent layer. And all the necessary constants have been evaluated on the basis of experimental data. So, we may now investigate some general solutions to this equation. This task will now occupy the rest of the paper.

5. RESULTS OF THE NUMERICAL INTEGRATIONS

Couette flow without pressure gradient

We shall study first a one-dimensional flow without pressure gradient. In this case all the fluid properties are functions of the single space dimension with a single free parameter, namely the level of turbulence inside the layer. The solutions were obtained numerically, as described in Section 2. In Fig. 5 the $k_+ \sim R$ relation is displayed. We can easily identify equation (3.4), as the horizontal line for a non-diffusional layer; equation (3.11) agrees very well with the laminar sublayer predictions, to the left of the figure, while equation (3.19) is seen to describe the upper right-hand side of

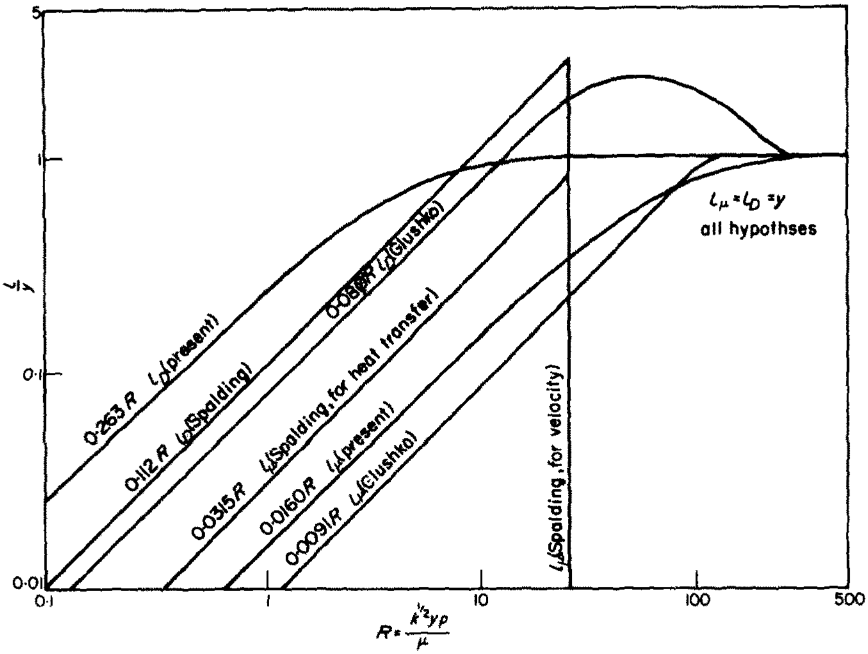

FIG. 4. Comparison of length scale distributions.

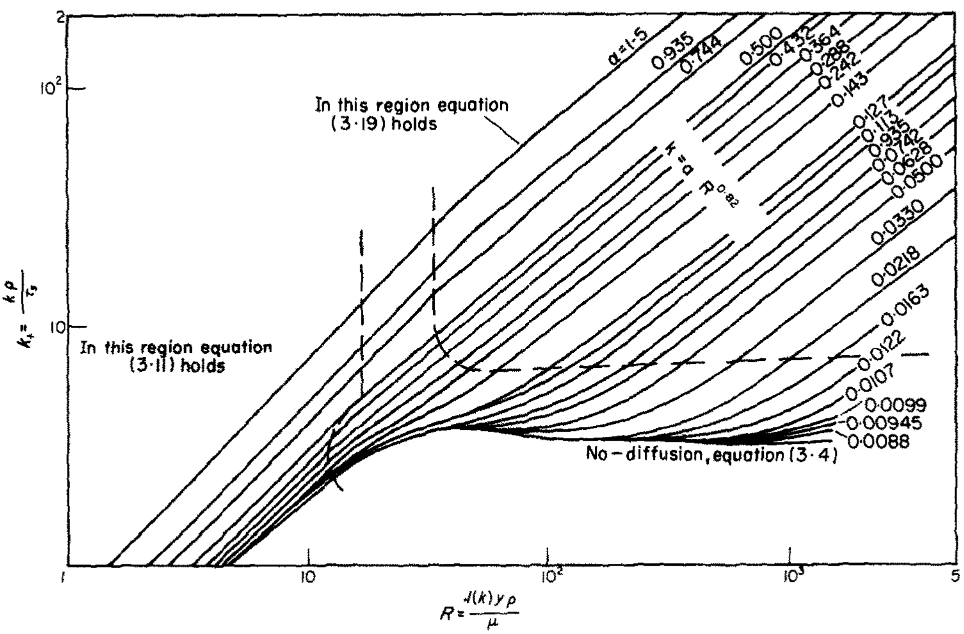

FIG. 5. $k_+ \sim R$ relation in a uniform-shear Couette flow.

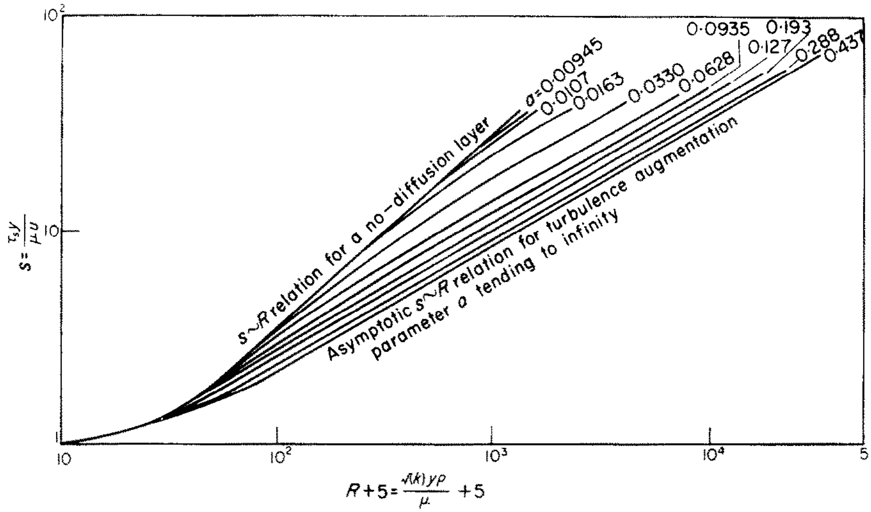

FIG. 6. $s \sim R$ relation in a uniform-shear Couette flow.

the figure. An interesting feature of the figure is that when the turbulence level in the layer increases, the flow approaches that of a no-generation layer, presumably because the turbulence generation vanishes, together with the velocity gradient.

We still have to attribute a convenient scale of the turbulence level to each of the lines in Fig. 5. We notice that apart from the single line describing the non-diffusional layer, each line is satisfying equation (3.19) in the upper right-hand side of the figure. Therefore, the constant in equation (3.19), now denoted "a", may serve as the scale of the degree of turbulence of the layer. We may now turn our attention to Fig. 6, where the $s \sim R$ relations are displayed for varying a. Again, the asymptotic solutions, equation (3.5), (3.14) and (3.21) are evident. The interesting

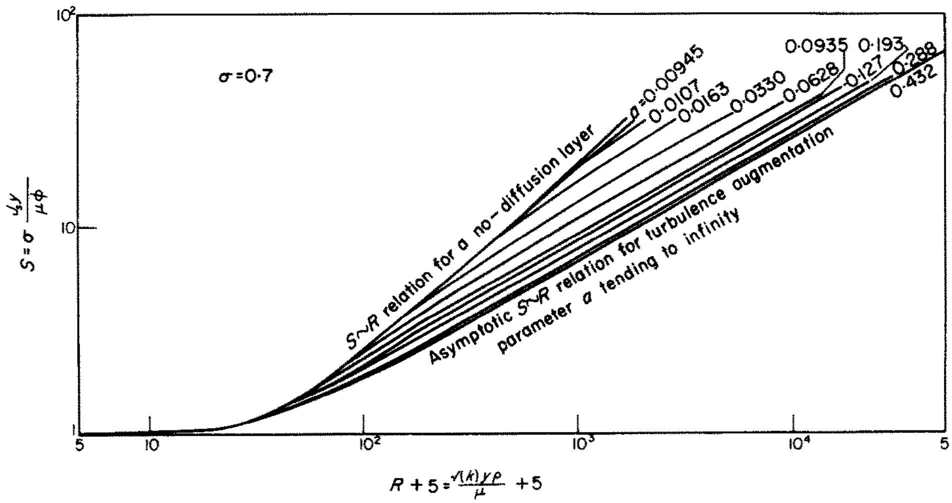

FIG. 7. $S \sim R$ relation in a uniform-shear Couette flow.

feature of this figure is the fact that when a is increased beyond, say, 0·5 the $s \sim R$ relation does not change any more.

In Fig. 7 the $S \sim R$ relation is shown for $\sigma = 0{\cdot}7$. The effect of increasing the Prandtl number is mainly to shift the $S \sim R$ lines to the left. In all other respects Fig. 7 is similar to Fig. 6.

A more conventional representation of the $s \sim R$ relation is in the form of $u_+ \sim y_+$ relation, shown in Fig. 8. As may be expected, an increased level of turbulence flattens the velocity profile.

Equilibrium Couette flows

The term equilibrium is used in the present section to describe a Couette flow in which the boundary condition in the outer edge of the flow has no influence on the flow. Such behaviour is possible in either of two cases: (i) when the boundary is very far away from the wall (in this case equilibrium will be maintained near the wall and, as we shall soon see, in the middle of the layer, but not near its outer edge); (ii) when an equilibrium boundary condition is specified (this amounts to the use of equation (3.7) as a boundary condition for the turbulence energy equation in very large distances from the wall). In order to demonstrate that an equilibrium flow does exist, various solutions for the same pressure gradient $p_+ = 0{\cdot}05$ were plotted in Fig. 9. It is clearly seen that as long as the boundary value of the turbulence energy is lower or slightly higher than the equilibrium value the turbulence energy profile approaches its equilibrium state quite rapidly. This is not true, however, for very high boundary values of the turbulence energy, which result in a turbulence energy profile higher than the equilibrium one. There is no equilibrium state for negative pressure gradients.

On the basis of the above discussion it is clear that, in an equilibrium flow, $k_{+,G}$ must be a unique function of $y_{+,G}$ and p_+. Thus, u_+ and

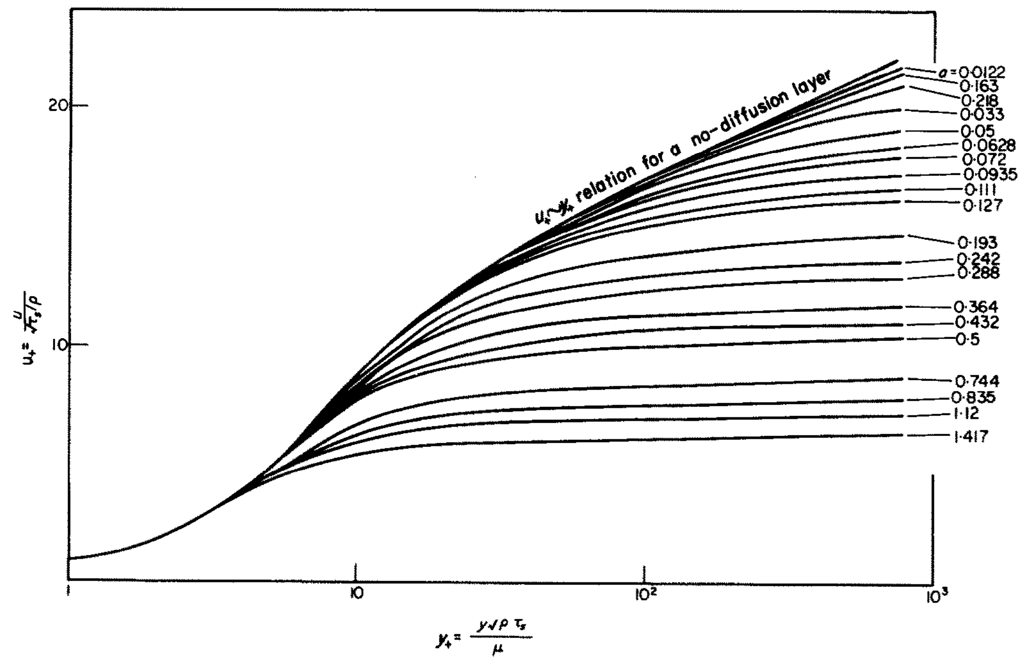

FIG. 8. $u_+ \sim y_+$ relation in a uniform-shear Couette flow.

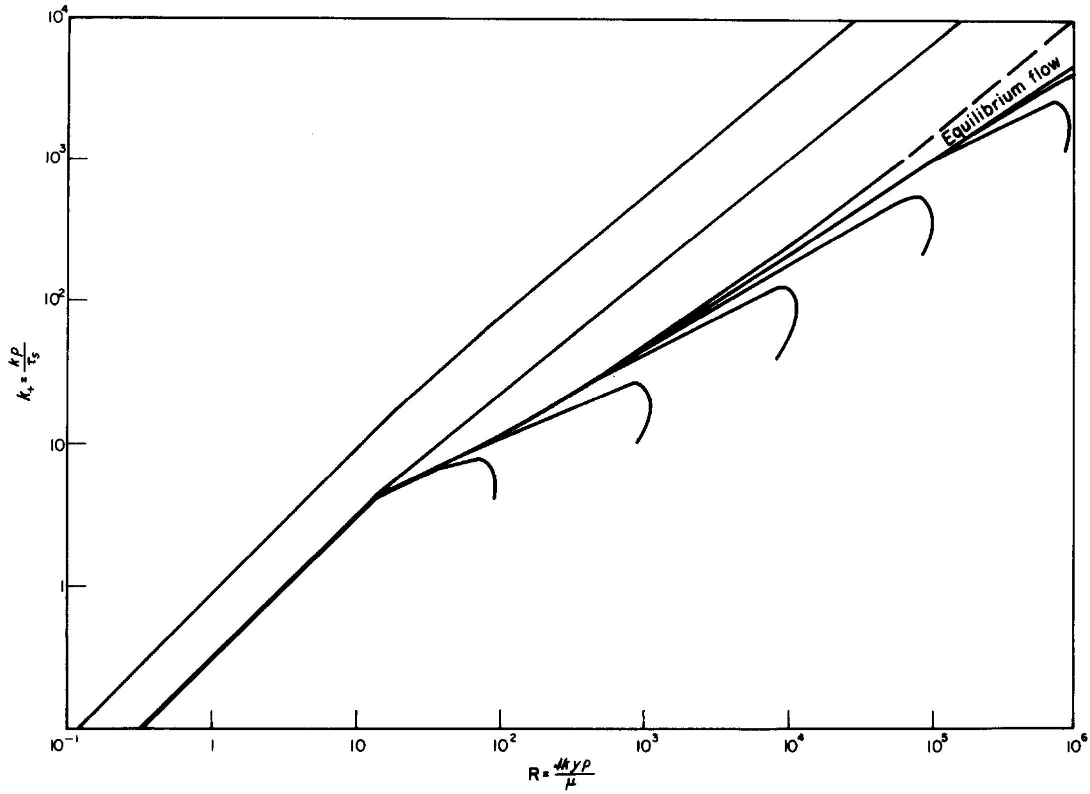

FIG. 9. $k_+ \sim R$ relation in equilibrium flow for $p_+ = 0{\cdot}05$.

k_+ are functions of y_+ and *only one free parameter*, p_+; φ_+ is then a function of y_+ and the two parameters p_+ and σ.

The equilibrium functions of k_+ and s are plotted in Figs. 10 and 11 as functions of R for various values of the parameter p_+. We note that when p_+ becomes large the flow may be described by the equations for the linear shear layer (3.7) and (3.8). No solutions are presented for a negative pressure gradient.

The equilibrium $S \sim R$ function for $\sigma = 0{\cdot}7$ is presented in Fig. 12. It has some similarity to Fig. 7, which displays the $S \sim R$ relation for a uniform-shear layer. This similarity is more apparent for small p_+. But even when p_+ is large the S values are not very different from those shown in Fig. 7 for corresponding values of turbulence, and the general asymptotic behaviour is similar.

The combined effect of pressure gradient and augmented turbulence

We must now distinguish between positive and negative pressure gradient. In the first case, if the level of turbulence in the outer edge of the layer is below the equilibrium one, most of the layer will become an equilibrium flow, with deviation from equilibrium only near the outer edge. If the level of turbulence in the outer edge is much higher than the equilibrium one, the flow will be identical with the one with a zero pressure gradient, but with turbulence augmentation.

When the pressure gradient is negative, we

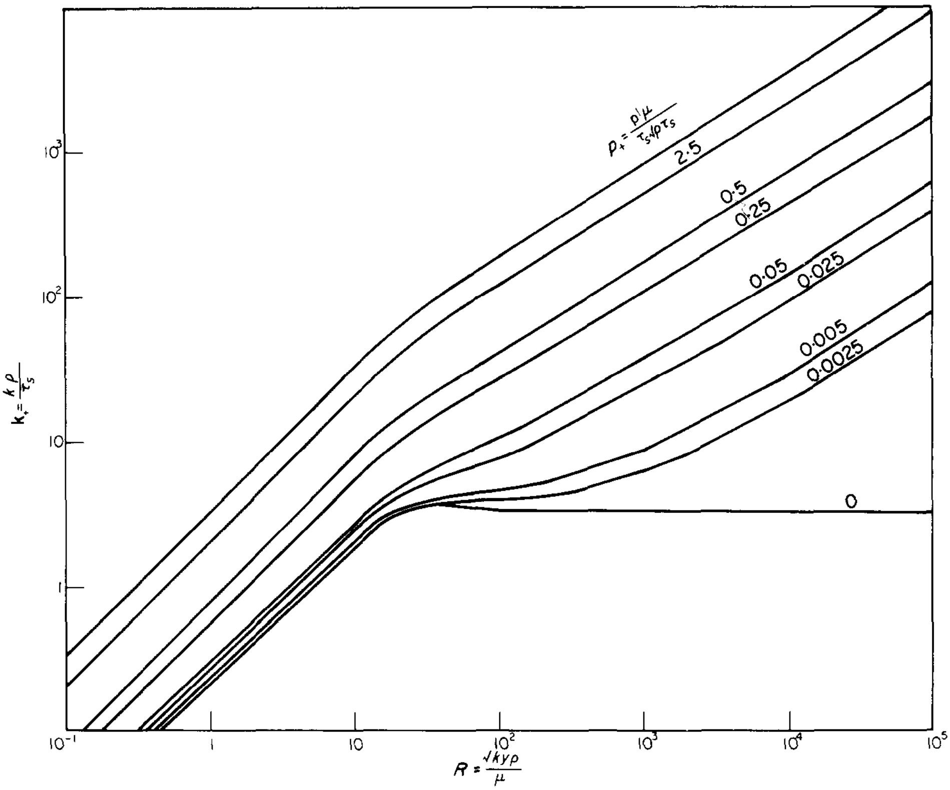

FIG. 10. $k_+ \sim R$ relation for equilibrium Couette flow.

concern ourselves only with that part of the layer where the shear stress is still positive. And in this part, again, the flow is always similar to the one without any pressure gradient, but with augmented turbulence.

Duct flow

The last solutions which we consider are for fully developed turbulent duct flow. It has been widely accepted that the flow near the walls of a duct is similar to a Couette flow. There is, however, a negative pressure gradient, and a diffusion of turbulence energy from the boundary layer, near the wall, towards the centre of the duct.

In the present problem there is only one parameter, namely the Reynolds number.† The pressure gradient is uniquely related to the skin friction (which is a function of the Reynolds number); and in a fully developed flow there is only one possible value of the turbulence energy on the centre-line.

The $k_+ \sim R$ and $s \sim R$ relations are plotted in Figs. 13 and 14 respectively, for various Reynolds numbers. Also plotted are some experimental data reported by Clark [14]. It will be noted, that near the wall, where say

$$\frac{p'y}{\tau_S} < 0{\cdot}2$$

† The Reynolds number will be based on the maximum velocity and duct half width.

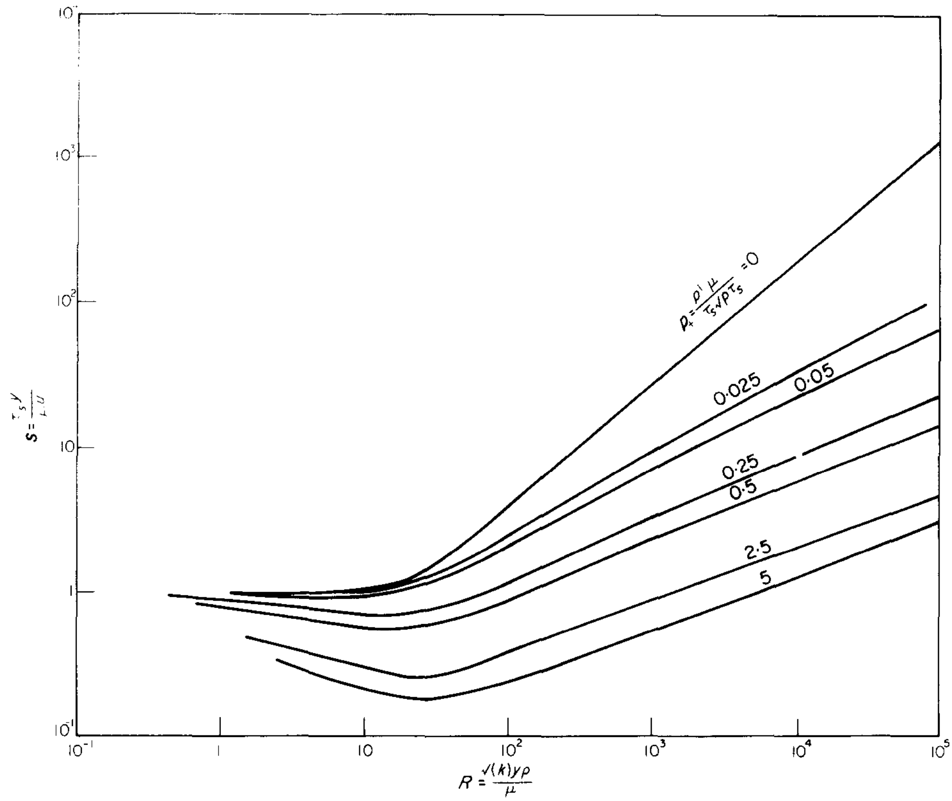

FIG. 11. $s \sim R$ relation for equilibrium Couette flow.

the flow is practically identical with a Couette flow without pressure gradient. In this region the present results are near to the experimental data, but not identical to it. However, if we recall the experimental difficulties which are present when measurements of u' and w' are performed by a hot wire anemomenter, we should not, perhaps, attempt a better agreement than has been achieved. Moreover, Clark's constants in the law of the wall are quite different from the ones used in the present paper, and this fact may be responsible for the deviations. The agreement in the outer region is reasonable.

6. DISCUSSION

The turbulence energy hypothesis. It has been shown that the present version of the hypothesis is general and flexible enough to enable predictions for a large variety of cases, without any special practices for the laminar sublayer. However, the accuracy of the predictions cannot be any better than that of the data used for constants fitting. At present the data is not reproducible. Even the constants $\varkappa$ and E in the logarithmic law of the wall tend to change from one experimental work to the other. Values of 0·4–0·44 have been suggested for $\varkappa$, and anything from 6 to 12 for E. It has been shown, however, that if we have the appropriate data, the theory may help us to screen it, and to explain some of the tendencies which are often found in such data. Moreover, the existence of this theory may promote some experimental work which may resolve some of the above difficulties.

The influence of turbulence augmentation. It

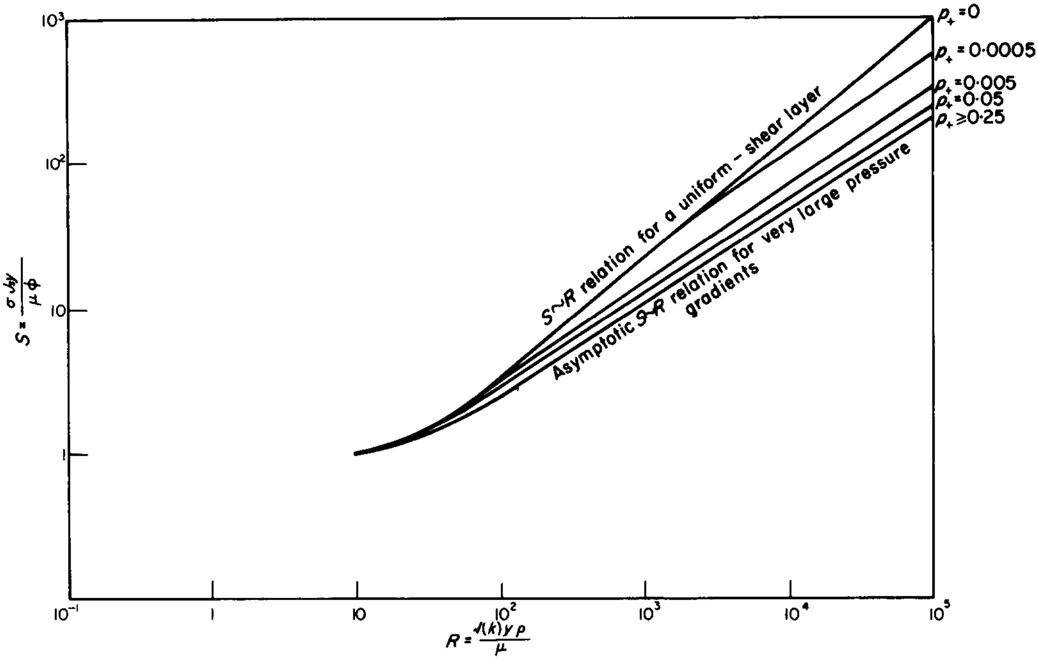

FIG. 12. $S \sim R$ relation for equilibrium Couette flow.

has been shown that turbulence augmentation has a very marked influence on the velocity and temperature distributions in the fully turbulent region of a Couette flow. However, the laminar sublayer remains unchanged until a fairly high rate of turbulence augmentation is reached.

An examination of Fig. 5, 6 and 7 reveals that the solutions for a zero pressure gradient may be correlated into algebraic relations quite easily. Such correlations have been obtained by the author [1], in connection with his work on two-dimensional flows. It is not necessary to present these correlations here, with

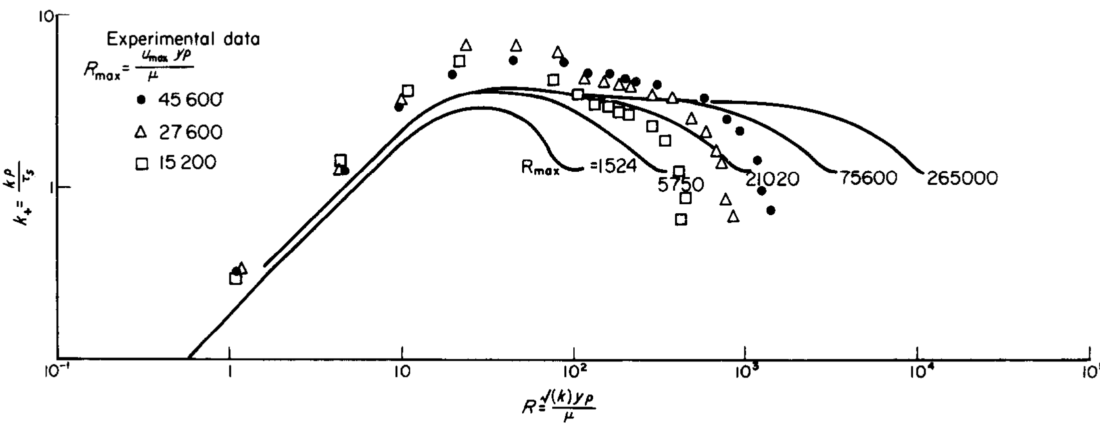

FIG. 13. $k_+ \sim R$ relation for a duct flow. Experimental data by Clark.

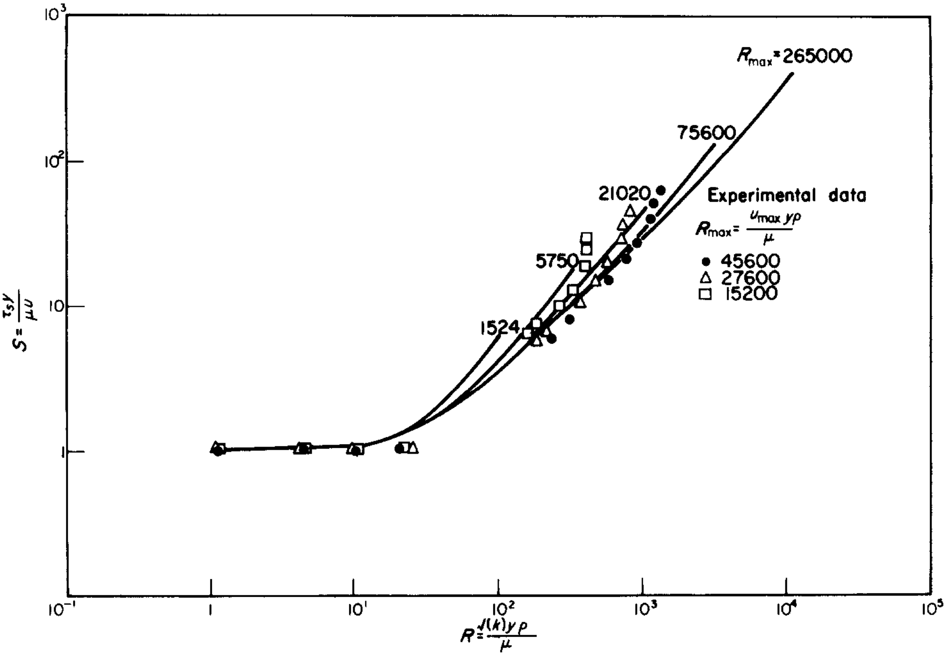

FIG. 14. $s \sim R$ relation for duct flow. Experimental data by Clark.

one exception: the $S(R, k_+)$ function in the fully turbulent region, and for $\sigma = 0{\cdot}71$, with turbulence level larger than that of the non-diffusional case, may be correlated by

$$S = 0{\cdot}139 y_+^{0{\cdot}58} k_+^{0{\cdot}29} \qquad (6.1)$$

in the vanishing-generation case, and

$$S = 0{\cdot}09 y_+^{0{\cdot}86} k_+^{0{\cdot}09} \qquad (6.2)$$

in the case of finite turbulence generation. These equations show that for small and medium turbulence augmentation S is almost independent of the level of turbulence. Kestin *et al.* [15] measured the influence of the level of turbulence on heat transfer in a zero-pressure gradient Couette flow with relatively low level of turbulence. They reported no influence of the level of turbulence on heat transfer in these conditions.

A note on experimental data. The scarcity of appropriate experimental data has already been mentioned. The existence of the present theory may, perhaps, stimulate further measurements of the kinetic energy and other fluctuating quantities. Such data will enable a better choice of the empirical constants to be made, and will also serve as a check on the suitability of the present hypothesis.

Some further theoretical work. The present model may be further developed as to include a differential equation for the length scales, and to account for non-uniform distribution of the Prandtl numbers, if new experimental data justifies such steps.

7. CONCLUSIONS

(1) The hypothesis, and in particular the present forms of the Reynolds stresses, the turbulence energy dissipation and the turbulent length scales seem to be in accord with the available experimental data for one-dimensional flows.

(2) The choice of k_+, s, S and R as variables instead of k_+, u_+, φ_+ and y_+ seems to be

justified, because it enables us to cope with cases of zero skin friction, and because it confines the results between limiting lines, thus reducing the spread of the results.

(3) The effect of turbulence augmentation on any flow is to reduce the turbulence generation. When the augmentation is sufficiently large the flow is identical to a zero shear one. In this case the $s \sim R$ and $S \sim R$ relation are not dependent on the turbulence augmentation any more.

(4) Adverse pressure gradient increases the turbulence level. The combined effect of the pressure gradient and of this increased turbulence is to lower the s values at a fixed R, without a limit. However, the combined effect on S, for a fixed R, is similar, qualitatively at least, to that of turbulence augmentation on a uniform shear flow.

(5) Flows with small favourable pressure gradient (when the shear stress is still positive) are very similar to uniform-shear flows.

(6) A low turbulence level in the outer boundary results in an equilibrium flow, appropriate to the pressure gradient in question.

(7) The existence of the present theory makes further measurements of turbulence energy and fluctuating quantities very desirable.

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Résumé—L'hypothèse de l'énergie turbulente de Kolgomorov–Prandtl est formulée d'une façon valable pour la sous-couche laminaire aussi bien que pour la région entièrement turbulente d'un écoulement unidimensionnel. Les constantes nécessaires sont ajustées aux données expérimentales disponibles. Des solutions numériques sont obtenues pour l'écoulement de Couette avec augmentation de la turbulence et gradient de pression et pour l'écoulement turbulent en conduite. On a obtenu un accord raisonnable avec les données expérimentales disponibles. Certains nouveaux groupes sans dimensions sont employés et l'on montre qu'ils sont supérieurs à ceux basés sur la vitesse de frottement. Les effets de l'augmentation de la turbulence et du gradient de pression sur les distributions de vitesse et de température sont étudiés. On trouve que les solutions tendent vers les solutions pour les cas limites. Les résultats sont portés sur quelques figures dans la cinquième section.

Zusammenfassung—Die Turbulenz-Energie-Hypothese von Kolomogorov-Prandtl ist so formuliert, dass sie für die laminare Unterschicht genauso gilt wie für den voll turbulenten Bereich einer eindimensionalen Strömung. Die erforderlichen Konstanten werden den verfügbaren Versuchsdaten angepasst. Numerische Lösungen werden für die Couette-Strömung mit Turbulenzanstieg und Druckgradient und für turbulente Rohrströmung erhalten. Zufriedenstellende Übereinstimmung mit verfügbaren experimentellen Werten wurde erreicht. Einige neu verwendete dimensionslose Gruppen erweisen sich jenen überlegen, die auf der Reibungsgeschwindigkeit beruhen. Die Einflüsse des Turbulenzanstiegs und des Druckgradienten auf die Geschwindigkeits- und Temperaturverteilung wurden untersucht. Es zeigte sich, dass sich die Lösungen Grenzfall-Lösungen nähern. Die Ergebnisse sind in Abbildungen im Abschnitt 5 wiedergegeben.

Аннотация—Гипотеза Колмогорова–Прандтля об энергии турбулентности сфомулирована в таком виде, что она справедлива для ламинарного подслоя также как и для полностью развитой турбулентной области плоского течения. Необходимые постоянные хорошо согласуются с имеющимися экспериментальными данными. Получены численные результаты для потока Куэтта при увеличении турбулентности и наличии градиента давления, а также для турбулентного течения в канале. Получено хорошее совпадение с имеющимися экспериментальными данными. Используются некоторые безразмерные группы и показано, что они выгодно отличаются от групп, основанных на скорости трения. Изучается влияние увеличения турбулентности и градента давления на распределение скорости и температуры. Найдено, что решения стремятся к решениям для предельных случаев. Результаты представлены графически.